Technical Report No. 32-497

Theory of Orbit Determination - Part I Classical Methods

John D. Anderson

T. W. Hamilton

Thomas W. Hamilton, Chief
Systems Analysis Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

October 1, 1963

Prepared Under Contract No. NAS 7-100
National Aeronautics & Space Administration

CONTENTS

ABSTRACT

A cursory review of the classical methods of orbit determination is given for the purpose of orienting the reader to the nature of the classical orbit determination problem. The Gaussian and Laplacian methods of obtaining a first approximation to the orbit are outlined, but no attempt is made to describe the computational procedures in detail. Instead, a list of references is included that provides exhaustive treatments of the classical orbit determination topics.

I. INTRODUCTION

Many volumes have been written on the classical methods of orbit determination. The subject has been developed through the efforts of some of the most prominent mathematicians, chiefly those who published in the second half of the eighteenth century. The problem of classical orbit determination is to reduce a limited number of direction observations, usually three, to a set of six orbital elements. The observed object may very well be newly discovered, the most common being a comet or minor planet. The purpose of the classical orbit determination methods is to provide a preliminary orbit so that the object can be recovered at a later date. Also, such an orbit can serve as a first approximation to a more refined determination that involves many observations over long periods of time. In either case, the determined orbital elements are quite satisfactory if they are based on simple, two-body Keplerian motion, and the determination of the orbit can be thought of as the search for a transformation that will convert three pairs of right ascension and declination measurements (six angles in all) to the six constants of the two-body motion.

It is fairly obvious that in space flight the classical methods will find limited application. On the one hand, a preliminary orbit is rarely required because the orbit of a space vehicle is not totally unknown, and a sufficient first approximation to the orbit can be obtained from the designed trajectory. On the other hand, angle observations are not the only types of data available when radio tracking is employed. Moreover, the classical methods do not adapt themselves directly to range and range-rate measurements. Therefore, in this report only an outline of the classical procedures is presented, and the reader is directed to the references for the details.

II. STATEMENT OF THE PROBLEM

Basically, if the transformation between angular observations and orbital elements exists then it can be written in the functional form

$$\left.\begin{aligned}\alpha_i &= \alpha_i\,(e_1, e_2, \cdots, e_6, t_i)\\ \delta_i &= \delta_i\,(e_1, e_2, \cdots, e_6, t_i)\end{aligned}\right\} i = 1,2,3 \qquad (1)$$

where $e_1, e_2, \cdots, e_6$ are six constants of the two-body motion and the subscript i refers to the i^{th} time of observation t_i. Eq. (1) states that if the six orbital elements $e_1, e_2, \cdots, e_6$ are known, then the position of the object in the sky can be determined as a function of the time t. This implies no more than the fact that a solution exists to the equations of motion and that the geometry of the observer in space is known. In mathematical terms, the solution to the equations of motion yields the position vector $\mathbf{r}$ of the object P with respect to some origin S, such as the center of the Sun or Earth (Fig. 1), and given the known position vector $\mathbf{R}$ of the observer O, the observer-centered or topocentric position vector $\boldsymbol{\rho}$ of the object can be determined by

$$\boldsymbol{\rho} = \mathbf{r} - \mathbf{R} \qquad (2)$$

The unit vector $\mathbf{L}$ that describes the direction of the object is

$$\mathbf{L} = \frac{\boldsymbol{\rho}}{\rho} \qquad (3)$$

where ρ is the magnitude of the vector $\boldsymbol{\rho}$. Now the components of $\mathbf{L}$ are the direction cosines of the object, and if the reference coordinate system is equatorial with the x-axis directed toward the vernal equinox, then the right ascension and declination can be determined from

$$\begin{aligned}L_x &= \cos\delta\cos\alpha\\ L_y &= \cos\delta\sin\alpha\\ L_z &= \sin\delta\end{aligned} \qquad (4)$$

Thus the transformation Eq. (1) does exist, and if the Jacobian of the system is not zero, it is theoretically possible to invert the six equations for the six unknown constants

$$\begin{aligned}e_j &= e_j\,(\alpha_1, \delta_1, t_1, \alpha_2, \delta_2, t_2, \alpha_3, \delta_3, t_3)\\ j &= 1, 2, \cdots, 6\end{aligned} \qquad (5)$$

The entire purpose of the classical orbit determination methods is to obtain this inverse, which, unfortunately, cannot be written in closed form. Thus, iterative techniques must be employed for the solution, and because the values of the e_j are often totally unknown, a first approximation to the solution is required. Finally, a process called differential correction modifies the first approximation until the values of e_j satisfy the system of Eq. (1). Again the reader is referred to the References (particularly Ref. 2 and 4) for a description of the differential correction methods. Also, these sources should be consulted for a detailed description of the computation of the first approximation. Only the basic ideas of two popular approximations, the Gaussian and Laplacian methods, are given here.

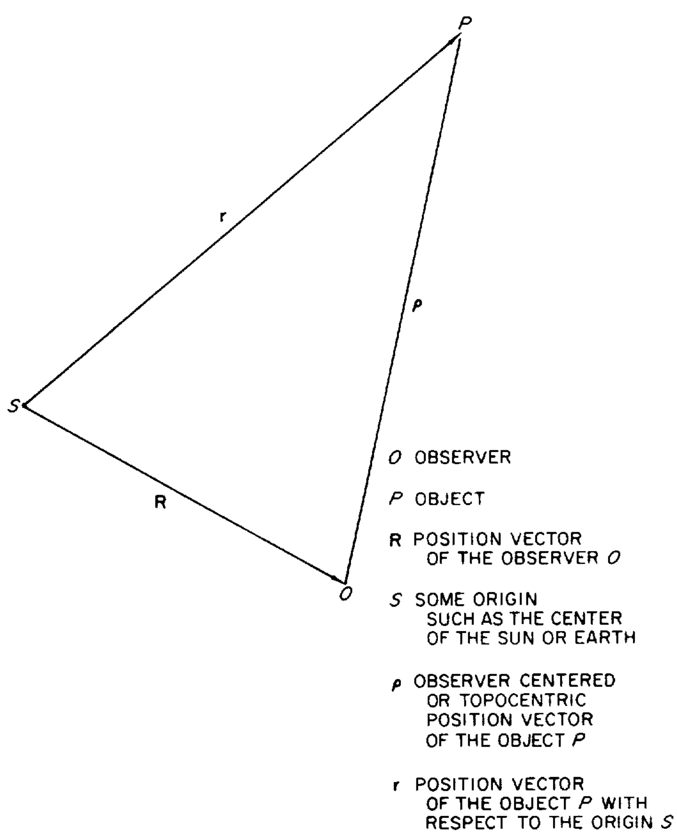

Fig. 1. Geometry of the observer and object in space

III. THE LAPLACIAN METHOD

The Laplacian method is based on writing Eq. (2) in the form

$$\mathbf{r} = \rho \mathbf{L} + \mathbf{R} \tag{6}$$

Actually, for comparison with the references, the vector **R** is oppositely directed in practically all the literature because traditionally the coordinates of the Sun, the most usual origin, are given with respect to the Earth. However, to avoid confusion with space flight practice, the vector **R** is directed toward the observer. In either case, the logic of the development is unaltered.

Now, the method of Laplace is based on twice differentiating Eq. (6) to yield two more equations for the velocity vector $\dot{\mathbf{r}}$ and acceleration $\ddot{\mathbf{r}}$. However, by assuming the validity of Newton's laws, the acceleration is given for two-body motion by

$$\ddot{\mathbf{r}} = -\frac{\mu \mathbf{r}}{r^3} \tag{7}$$

where r is the magnitude of $\mathbf{r}$ and μ is the usual constant of proportionality. Therefore, the acceleration can be eliminated from the two additional equations leaving only the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ components of the object P.

$$\dot{\mathbf{r}} = \dot{\rho}\mathbf{L} + \rho\dot{\mathbf{L}} + \dot{\mathbf{R}} \tag{8}$$

$$-\frac{\mu \mathbf{r}}{r^3} = \ddot{\rho}\mathbf{L} + 2\dot{\rho}\dot{\mathbf{L}} + \rho\ddot{\mathbf{L}} + \ddot{\mathbf{R}} \tag{9}$$

The position vector $\mathbf{r}$ can be eliminated in Eq. (9) by using Eq. (6), and after rearranging terms there results

$$\left(\ddot{\mathbf{L}} + \frac{\mu \mathbf{L}}{r^3}\right)\rho + 2\dot{\mathbf{L}}\dot{\rho} + \mathbf{L}\ddot{\rho} = -\left(\ddot{\mathbf{R}} + \frac{\mu \mathbf{R}}{r^3}\right) \tag{10}$$

Because Eq. (10) is a vector equation there are actually three equations available for each time of observation. Consider these equations at a particular time t_2. Now the vectors **L** and **R** are both known at this time, and if their derivatives could also be determined, then Eq. (10) would represent three equations in three unknowns, ρ, $\dot{\rho}$, and $\ddot{\rho}$. In addition, if the radius r were given, the three equations would be linear, and their solution would follow quite easily. In fact, the linear solution is obtained for ρ and $\dot{\rho}$ by the method of determinants. With the components of **R** given by X, Y, and Z there results

$$\rho = \frac{\begin{vmatrix} -\left(\ddot{X} + \frac{\mu X}{r^3}\right) & 2\dot{L}_x & L_x \\ -\left(\ddot{Y} + \frac{\mu Y}{r^3}\right) & 2\dot{L}_y & L_y \\ -\left(\ddot{Z} + \frac{\mu Z}{r^3}\right) & 2\dot{L}_z & L_z \end{vmatrix}}{\begin{vmatrix} \left(\ddot{L}_x + \frac{\mu L_x}{r^3}\right) & 2\dot{L}_x & L_x \\ \left(\ddot{L}_y + \frac{\mu L_y}{r^3}\right) & 2\dot{L}_y & L_y \\ \left(\ddot{L}_z + \frac{\mu L_z}{r^3}\right) & 2\dot{L}_z & L_z \end{vmatrix}} \tag{11}$$

or by separating out the $1/r^3$ terms, Eq. (11) can be written in the notation of Herrick as

$$D\rho = A' - B'/r^3 \tag{12}$$

and similarly for $\dot{\rho}$

$$D\dot{\rho} = C' - E'/r^3 \tag{13}$$

where

$$A' = \begin{vmatrix} -\ddot{X} & 2\dot{L}_x & L_x \\ -\ddot{Y} & 2\dot{L}_y & L_y \\ -\ddot{Z} & 2\dot{L}_z & L_z \end{vmatrix}$$

$$B' = \mu \begin{vmatrix} X & -2\dot{L}_x & -L_x \\ Y & -2\dot{L}_y & -L_y \\ Z & -2\dot{L}_z & -L_z \end{vmatrix}$$

$$C' = \begin{vmatrix} \ddot{L}_x & -\ddot{X} & L_x \\ \ddot{L}_y & -\ddot{Y} & L_y \\ \ddot{L}_z & -\ddot{Z} & L_z \end{vmatrix}$$

$$E' = -\mu \begin{vmatrix} \ddot{L}_x & -X & L_x \\ \ddot{L}_y & -Y & L_y \\ \ddot{L}_z & -Z & L_z \end{vmatrix}$$

Still assuming that the derivatives of **L** and **R** are available, the method of Laplace finds a value of r that will satisfy both Eq. (12) and Eq. (6), or, more rigorously, instead of Eq. (6) itself, the so-called triangle equation is used

$$r^2 = \mathbf{r} \cdot \mathbf{r} = \rho^2 + R^2 + 2\rho\,(\mathbf{R} \cdot \mathbf{L}) \tag{14}$$

The simultaneous solution of Eq. (12) and (14) is not described here, but all that is involved is an iterative solution to Eq. (14) by means of a Newton-Raphson procedure.

With the iterative solution to r available, it is now possible to return to Eq. (13) and compute the range rate $\dot{\rho}$. Finally, Eq. (6) and (8) provide the position and velocity which can serve as orbital elements, or, alternatively, for the case where the classical Keplerian elements are required, the usual transformations from position and velocity to orbital elements can be applied. It should be remembered that, so far, all the above manipulations have been performed at the midpoint time t_2, and the first and second derivatives of **R** and **L** have been assumed known. Actually, the other two points t_1 and t_3, with the point t_2, serve to provide these derivatives through numerical differentiation, and thus, as expected, all three points enter into the determination of r and $\dot{r}$ at t_2. The differentiation formula is that of Poincaré and is derived from a second-order Taylor's formula

$$\mathbf{R}_i = \mathbf{R}_2 + \dot{\mathbf{R}}_2\,\tau_i + \frac{1}{2}\ddot{\mathbf{R}}_2\,\tau_i^2 \qquad i = 1, 3 \tag{15}$$

where

$$\tau_i = t_i - t_2 \tag{16}$$

Of course, the vector **R** can be replaced by **L** in Eq. (15) and the linear solution for $\dot{\mathbf{L}}_2$ and $\ddot{\mathbf{L}}_2$ is exactly the same as for $\dot{\mathbf{R}}_2$ and $\ddot{\mathbf{R}}_2$.

$$\tau_1\,\tau_3\,(\tau_1 - \tau_3)\,\dot{\mathbf{R}}_2 = -\tau_3^2\,\mathbf{R}_1 + (\tau_3^2 - \tau_1^2)\,\mathbf{R}_2 + \tau_1^2\,\mathbf{R}_3 \tag{17}$$

$$\tau_1\,\tau_3\,(\tau_1 - \tau_3)\,\ddot{\mathbf{R}}_2 = 2\tau_3\,\mathbf{R}_1 + 2\,(\tau_1 - \tau_3)\,\mathbf{R}_2 - 2\tau_1\,\mathbf{R}_3 \tag{18}$$

IV. THE GAUSSIAN METHOD

The second technique for computing a first approximation is often called Gauss's method, although the formulation of many of the ideas was described by Lagrange, and the computational form of the method used today reflects modifications by Gibbs, Herrick and others. The basic equations give position and velocity at the middle time t_2 in terms of position at the three times of observation. Again, in the notation of Herrick

$$\mathbf{r}_2 = c_1\mathbf{r}_1 + c_3\mathbf{r}_3 \tag{19}$$

$$\dot{\mathbf{r}}_2 = -d_1\mathbf{r}_1 + d_2\mathbf{r}_2 + d_3\mathbf{r}_3 \tag{20}$$

Equation (19) simply states that the three position vectors of the object lie in the same plane, or, in accordance with Kepler's first law, that the motion of the object is planar. The second equation is a numerical differentiation formula similar to Eq. (17) in the Laplacian method. The c and d coefficients, as given by Herrick, are listed below without their derivation which is readily available (Ref. 1 and 4). The definition of τ_i is the same as in Eq. (16)

$$c_i = A_i + \frac{B_i'}{r_2^3}\left(1 + \frac{B_2}{r_2^3}\right) \qquad i = 1,3 \tag{21}$$

where

$$A_1 = \frac{\tau_3}{\tau_3 - \tau_1} \qquad A_3 = -\frac{\tau_1}{\tau_3 - \tau_1}$$

$$B_1' = A_1(B_1 + B_2) \qquad B_3' = A_3(B_3 + B_2)$$

and

$$B_1 = \frac{\mu}{12}(\tau_1^2 - \tau_1\tau_3 - \tau_3^2)$$

$$B_2 = \frac{\mu}{12}(\tau_1^2 - 3\tau_1\tau_3 - \tau_3^2)$$

$$B_3 = \frac{\mu}{12}(\tau_1^2 - \tau_1\tau_3 + \tau_3^2)$$

$$d_i = G_i + \frac{H_i}{r_i^3} \qquad i = 1,2,3 \tag{22}$$

where

$$G_1 = \frac{\tau_3^2}{\tau_1\tau_3(\tau_1 - \tau_3)} \qquad G_3 = \frac{\tau_1^2}{\tau_1\tau_3(\tau_1 - \tau_3)}$$

$$H_1 = \frac{\mu\tau_3}{12} \qquad H_3 = -\frac{\mu\tau_1}{12}$$

$$G_2 = G_1 - G_3 \qquad H_2 = H_1 - H_3$$

note that if the term H_i/r_i^3 is neglected in d_i, then Eq. (20) reduces to Eq. (17).

Now the Gaussian solution makes use of Eq. (19) by substituting Eq. (6) for the position vector into Eq. (19) and by grouping terms as follows:

$$(C_1\rho_1)\mathbf{L}_1 - \rho_2\mathbf{L}_2 + (C_3\rho_3)\mathbf{L}_3 = -C_1\mathbf{R}_1 + \mathbf{R}_2 - C_3\mathbf{R}_3 \tag{23}$$

a linear solution to ρ_2 is now obtained in the form

$$E\rho_2 = F_1C_1 - F_2 + F_3C_3 \tag{24}$$

where

$$E = -\begin{vmatrix} L_{x1} & L_{x2} & L_{x3} \\ L_{y1} & L_{y2} & L_{y3} \\ L_{z1} & L_{z2} & L_{z3} \end{vmatrix}$$

and

$$F_i = -\begin{vmatrix} L_{x1} & X_i & L_{x3} \\ L_{y1} & Y_i & L_{y3} \\ L_{z1} & Z_i & L_{z3} \end{vmatrix} \quad i = 1,2,3$$

The scalar quantity E is recognized as the triple scalar product

$$E = \mathbf{L}_1 \cdot (\mathbf{L}_3 \times \mathbf{L}_2)$$

which must remain nonzero if the solution to ρ_2 is to exist. Geometrically this says that the three observation vectors must not be coplanar, or, equivalently, that if the three observations lie on a great circle arc in the sky then the solution is indeterminate. Procedures for dealing with this type of singularity are described in the literature. However, if E is nonzero the solution is simply one of recognizing that the only unknowns in Eq. (24) are ρ_2 and r_2 and that in a manner analogous to the Laplacian method, Eq. (24) can be solved iteratively with the triangle Eq. (14) to obtain both ρ_2 and r_2. All that remains is to evaluate the vectors $\mathbf{r}_2$ and $\dot{\mathbf{r}}_2$ by the following process:

a. Solve Eq. (23) for $C_1\rho_1$ and $C_3\rho_3$ in a fashion similar to Eq. (24). This is usually done simultaneously with the solution for ρ_2.

b. Using the value of r_2 from the simultaneous solution of Eq. (24) and (14), compute C_1 and C_3 and obtain ρ_1 and ρ_3 by $\rho_i = (C_i\rho_i)/C_i$.

c. Compute the position vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ by Eq. (6).

b. Compute d_1 and d_3 with $r_i^2 = \mathbf{r}_i \cdot \mathbf{r}_i$, and, finally, use Eq. (19) and (20) to obtain the required $\mathbf{r}_2$ and $\dot{\mathbf{r}}_2$.

REFERENCES

1. Baker, R. M. L., Jr., and Makemson, M. W., *An Introduction to Astrodynamics,* Academic Press, New York, 1960, Chapter 5.
2. Brouwer, D., and Clemence, D. M., *Methods of Celestial Mechanics,* Academic Press, New York, 1961, Chapter 9.
3. Dubyago, A. D., *The Determination of Orbits,* The Macmillan Co., New York, 1961.
4. Herrick, S., *Astrodynamics,* Van Nostrand, Princeton, N.J., In Press.
5. Moulton, F. R., *An Introduction to Celestial Mechanics,* Macmillan, New York, 1914, Chaper 6.